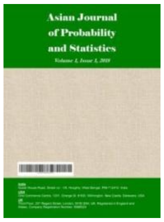

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Efficiency of Bayesian Approaches in Quantile Regression with Small Sample Size

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

Quantile regression is a statistical technique intended to estimate, and conduct inference about the conditional quantile functions. Just as the classical linear regression methods estimate model for the conditional mean function, quantile regression offers a mechanism for estimating models for the conditional median function, and the full range of other conditional quantile functions. In the Bayesian approach to variable selection prior distributions representing the subjective beliefs about the parameters are assigned to the regression coefficients. The estimation of parameters and the selection of the best subset of variables is accomplished by using adaptive lasso quantile regression. In this paper we describe, compare, and apply the two suggested Bayesian approaches. The two suggested Bayesian suggested approaches are used to select the best subset of variables and estimate the parameters of the quantile regression equation when small sample sizes are used. Simulations show that the proposed approaches are very competitive in terms of variable selection, estimation accuracy and efficient when small sample sizes are used.

Keywords: Quantile regression; small sample size; selection of variables; estimated risk; relative estimated risk; Bayesian approaches.

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1. INTRODUCTION

Quantile regression [1] has gained increasing popularity as it provides richer information than the classic mean regression. Quantile regression is a statistical technique intended to estimate, and conduct inference about the conditional quantile functions. Just as the classical linear regression methods estimate models for the conditional mean function, quantile regression offers a mechanism for estimating models for the conditional median function and the full range of other conditional quantile functions. [2] proposed an efficient algorithm that computes the entire solution path of the lasso regularized quantile regression. [3] focus on the variable selection aspect of penalized quantile regression. Under some mild conditions, this study demonstrates the oracle properties of the SCAD (smoothly clipped absolute deviation) and adaptive-lasso penalized. [4] consider quantile regression in high-dimensional sparse models. In such models, the overall number of regressors is very large, possibly much larger than the sample size. [5] proposed the composite quantile regression estimator by averaging quantile regressions. This study illustrates that the composite quantile regression is selection consistent and can be more robust in various circumstances.

Hierarchical Bayesian modeling provides a flexible and interpretable way of extending simple models by incorporating heuristic processes. Bayesian statistics provides a compelling and influential framework for representing and processing information. Over the last few decades, it has become a major approach in the field of statistics, and has come to be accepted in many or most of the physical, biological and human sciences. Bayesian models have great flexibility, and one aspect is the use of hierarchical structures in the model. Hierarchical models are normally used when there are a large number of similar units (loci, populations, individuals, etc.) and there is uncertainty of whether these units should be parameterized identically or independently. Bayesian inference, combined with Markov Chain Monte Carlo (MCMC) algorithms, has become increasingly popular and Bayesian approaches to quantile regression have been developed by [6]. The two major advantages of Bayesian inference for quantile regression models, as compared to the classical methods, are that (i) it does not rely on approximations to the asymptotic variances of the estimators, and (ii) it provides estimation and forecasts which fully take into account parameter uncertainty. [7] illustrates the application of Bayesian inference to quantile regression. Bayesian inference regards unknown parameters as random variables and describes an MCMC algorithm to estimate the posterior densities of quantile regression parameters. [8] extended this work to analyzing a Tobit quantile regression model, a form of the censored model in which $y_i = y_i^*$ is observed if $y_i^* > 0$ and $y_i = 0$ is observed otherwise. A regression model then relates the unobserved y_i to the covariants x_i. [9] used the asymmetric Laplace (AL) likelihood and combine MCMC with the expectation maximizing (EM) algorithm, to determine inference on quantile regression for longitudinal data. [10] used the AL likelihood combined with non-parametric regression modeling using piecewise polynomials to implement automatic curve fitting for quantile regression. Recently, both [11,12] proposed a correction to the MCMC iterations to construct asymptotically valid intervals. [13] used the same approach, but incorporating natural cubic splines. [14] pointed out that the value of quantile τ not only controls the quantile but also the skewness of the AL distribution, resulting in limited explicitly. [15] using the R package bayes QR for Bayesian estimation of quantile regression. The package contains methods for both continuous as well as binary dependent variables. The residual distribution is symmetric when modeling the median. This motivated [16,17] to consider a more flexible residual distribution constructed using a Dirichlet process prior but still having the quantile equal to 0. The analysis of [14] included a general scale mixture of AL densities with skewness τ in their analysis, but conclude that in terms of ability to predict new observations, a general mixture of uniform distributions performs the best.

The organization of this paper is as follows: In Section 2 the study described Bayesian lasso quantile regression methods which were used in this study. In section 3 we introduce two new approximation methods to solve full condition posterior distribution. The first method is based on the Gamma function and the second method depend on the important sample. The simulation study for some distribution are given in Section 4. Finally, discussion and concluding analysis is provided in Section 5.

2. BAYESIAN LASSO QUANTILE REGRESSION

[17] employed a Laplacian prior $p\left(\beta_j \middle| \sigma, \lambda\right)$ = $(\sigma\lambda/\ )\exp\left\{-\sigma\lambda\left|\beta_j\right|\right\}$ on β_j , $\beta_j \in \beta$, where β is

parameter of quantile regression and assumed that the residuals ϵ become from the skewed Laplacian distribution. Specifically, Laplacian prior distributions are placed on the regression coefficients. [18] extended this idea to Bayesian adaptive Lasso quantile regression (BALQR). They put different penalization parameters on the different regression coefficients. Thus, we propose a Laplacian prior on β_j taking the form

$$p(\beta_j \mid \sigma,\lambda_j) = -\frac{\sigma^{\frac{1}{2}}}{2\lambda_j}\exp\left\{-\frac{\sigma^{\frac{1}{2}|\beta_j|}}{\lambda_j}\right\}, \quad -\infty < \beta_j < \infty \tag{2.1}$$

where λ_j is a nonnegative regularization parameter and σ is the scale parameter. [19,20] treated the hyperparameters of the inverse Gamma prior, with unknown parameters and estimated them along with the other parameters, which can be represented as a scale mixture of normal with an exponential mixing density [21].

$$\frac{\nu}{2}\exp\{\nu \mid t \mid\} = \int_0^\infty \frac{1}{\sqrt{2\pi s}}\exp\{-t^2/2s\}\frac{\nu^2}{2}\exp\left\{-\frac{\nu^2 s}{2}\right\}ds, \qquad \nu > 0 \tag{2.2}$$

Let, $\nu_j = \sigma^{\frac{1}{2}}/\lambda_j$ Then the proposed Gamma distribution prior can be written as

$$\begin{aligned} p(\beta_j \mid S_j) &= \frac{\nu^2}{2}\exp\{-\nu_j \mid \beta_j \mid\} \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi s_j}}\exp\{-\beta_j^2/2s_j\}\frac{\nu_j^2}{2}\exp\left\{-\frac{\nu_j^2 s_j}{2}\right\}ds_j \end{aligned} \tag{2.3}$$

then,

$$p(\beta_j \mid S_j) = \int_0^\infty \frac{1}{\sqrt{2\pi s_j}}\exp\{-\beta_j^2/2s_j\}\frac{\sigma}{2\lambda_j^2}\exp\left\{-\frac{\sigma s_j}{2\lambda_j^2}\right\}ds_j \tag{2.4}$$

where the inverse Gamma distribution priors on λ_j^2 (not λ_j) is of the form

$$p(\lambda_j^2 \mid \delta,\tau) = \frac{\tau^\delta}{\Gamma(\delta)}(\lambda_j^2)^{-1-\delta}\exp\left\{-\frac{\tau}{\lambda_j^2}\right\}, \quad \lambda_j > 0 \tag{2.5}$$

where $\delta > 0$ and $\tau > 0$ are two hyper-parameters. The posterior density function of λ_j^2, combining the prior Equation (2-4) with Equation (2-5), is the inverse of Gamma distribution with the shape parameter $1+\delta$ and the rate parameter $\sigma s_j/2+\tau$ .The amount of shrinkage in the prior Equation (2-5) depends on the values of the hyper-parameters τ and σ . Because smaller the τ and larger σ lead to bigger penalization, it is important to treat as unknown parameters to avoid enforcing specific values that affect the estimates of the regression coefficients. This procedure is quite different from Bayesian lasso quantile regression reported in [17]. BALQR uses a Laplacian prior for β_j such that each β_j has Lasso-type of penalization parameter $\sigma^{\frac{1}{2}}/\lambda_j$, as in the adaptive Lasso.

The BALQR is a Bayesian hierarchical model given by:

$$\underline{y} = \beta_0 + \underline{x}'\underline{\beta} + \theta\underline{z} + \phi\xi_i\sqrt{\sigma^{-1}\underline{z}} \tag{2.6}$$

where

$\underline{\boldsymbol{Y}} = (Y_1, Y_2, Y_3, \dots, Y_n), \underline{\boldsymbol{X}} = (X, X_2, X_3, \dots, X_n)\underline{\boldsymbol{Z}} = (Z_1, Z_2, Z_3, \dots, Z_n),\ \underline{\boldsymbol{\beta}} = (\beta_1, \beta_2, \beta_3, \dots, \beta_n)$

We consider that β_{0j} follows a standard normal distribution (0,1) with probability density function (pdf) as follows

$$P(\beta_{0j}) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{\beta_{0j}^2}{2}), \qquad -\infty < \beta_{0j} < \infty. \tag{2.7}$$

The ξ_i follows a standard normal distribution (0,1) with pdf as follows

$$P(\xi_i) = \frac{1}{\sqrt{2\pi}}\exp-\frac{\xi_i^2}{2}, \qquad -\infty < \xi_i < \infty. \tag{2.8}$$

The z_i follows an exponential with parameter σ with pdf as follows

$$P(z_i \mid \sigma) = \sigma\exp(-\sigma z_i), \qquad \sigma > 0. \tag{2.9}$$

However, β, s follows the joint pdf composed standard normal distribution with parameters (0,1) times exponential with parameter $\frac{\sigma}{2\lambda_j^2}$

$$P(\beta_j, s_j \mid \sigma, \lambda_j^2) = \frac{1}{\sqrt{2\pi s_j}}\exp\{-\frac{\beta_j^2}{2s_j}\}\frac{\sigma}{2\lambda_j^2}\exp(-\frac{\sigma s_j}{2\lambda_j^2}),\ \ \lambda_j > 0,\ \ \sigma > 0 \tag{2.10}$$

where , j = 1,2,…, k

λ follows an inverse Gamma distribution with shape parameter $(1+\delta)$, which is given in (2-9)

б follows a Gamma distribution with two parameters (a,b) as follows

$$P(\sigma) \propto \sigma^{a-1} \exp(-b\sigma), \qquad \sigma > 0. \qquad (2.11)$$

τ and δ have a joint non-informative prior distribution as follows

$$P(\tau,\delta) \propto \tau^{-1}, \qquad \tau > 0. \qquad (2.12)$$

The joint posterior distribution of all parameters is given by:

$$P\left(\beta_{0j}, \underline{\beta}, \underline{z}, \underline{s}, \sigma, \lambda_1, \dots, \lambda_k \middle| \underline{Y}, \underline{X}\right)$$
$$\propto P(\underline{Y}|\beta_{0j}, \underline{\beta}, \underline{z}, \sigma, \underline{X}) \prod_{i=1}^{n} P(\underline{z}|\sigma)$$
$$\times \prod_{j=1}^{k} P(\beta_j, s_j|\sigma, \lambda_j^2) P(\lambda_j^2|\delta,\tau) P(\sigma) P(\tau,\delta)$$

$$\propto \prod_{i=1}^{n} \frac{\sigma}{\sqrt{\sigma^{-1}\phi^2 z_i}} \exp\{\frac{-\sigma(y_i - \beta_{0j} - x_i^{'}\beta - \theta z_i)^2}{2\phi^2 z_i} - \sigma z_i\}$$
$$\times \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi s_j}} \exp\{-\frac{\beta_j^2}{2s_j}\} \frac{\sigma}{2\lambda_j^2} \exp(-\frac{\sigma s_j}{2\lambda_j^2}) \frac{\tau^{\delta}}{\Gamma(\delta)} (\lambda_j^2)^{-1-\delta} \exp(-\frac{\tau}{\lambda_j^2})$$
$$\times \tau^{-1} \sigma^{a-1} \exp(-b\sigma). \qquad (2.13)$$

Equation (2.13) yields a tractable and efficient Gibbs sampler that works as follows:

1.Generate z_i , x_i and y_i from N(0,1).

2. Fix the value of p , so that the p^{th} quantile is modeled.

3. Simulate $\beta_0 | . \sim N(\overline{\beta}_0, s^2_{\beta_0})$, where $\overline{\beta}_0 = (n^{-1}) \sum_{i=1}^{n} (y_i - x_i^{'}\beta - \theta z_i)$ and $s^2_{\beta_0} = \frac{\sigma\phi^2}{n^2} \sum_{i=1}^{n} \mathbf{z}_i$.

4. Simulate $\mathbf{z}_i^{-1} | . \sim$ Inverse Gaussian distribution $(\mu^{'}, \lambda^{'}),\ i = 1,2,...,n$ where

$$\mu^{'} = \sqrt{\frac{\theta^2 + 2\phi^2}{(y_i - x_i^{'}\boldsymbol{\beta})^2}},\ \lambda^{'} = \frac{\sigma(\theta^2 + 2\phi^2)}{\phi^2} \qquad (2.14)$$

in the parametrization of inverse Gaussian distribution the density function is given by

$$f(x | \lambda^{'}, \mu^{'}) = \sqrt{\frac{\lambda^{'}}{2\pi}} x^{\frac{-3}{2}} \exp\{-\lambda^{'} \frac{(x - \mu^{'})^2}{2(\mu^{'})^2 x}\}, \quad x > 0 \qquad (2.15)$$

5. Simulate $\beta_j | . \sim$ normal distribution $N(\overline{\beta}_j, \widetilde{\sigma}_j^2),$

where

$$\widetilde{\sigma}_j^{-2} = \sigma\phi^{-2} \sum_{i=1}^{n} x_{ij}^2\ z_i^{-1} + s_j^{-1} \qquad (2.16)$$

and

$$\overline{\beta}_j = \widetilde{\sigma}_j^2\ \sigma\ \phi^{-2} \sum_{i=1}^{n} x_{ij}^2\ z_i^{-1} (y_i - \beta_0 - \sum_{l \neq j}^{n} x_{il}\ \boldsymbol{\beta} - \boldsymbol{\theta}\mathbf{z}_i), \qquad (2.17)$$

6. Simulate $s_j | . \sim$ Inverse Gaussian distribution$(\mu', \lambda'), i = 1,2,\dots k$ where

$$\mu' = \sqrt{\frac{\beta_j^2\ \lambda_j^2}{\sigma}},\ \lambda' = \beta_j^2. \qquad (2.18)$$

7. Simulate $\sigma | . \sim$ Gamma distribution (a_1, a_2) , where

$$a_1 = \frac{3n}{2} + k + a \qquad (2.19)$$

Where n is sample size, k is number of parameter, a is Gamma parameter in the prior distribution in equation (2-11) and

$$a_2 = \sum_{i=1}^{n} \frac{(y_i - x_i^{'}\beta - \theta z_i)^2}{2\phi^2\ z_i} + z_i + \sum_{j=1}^{n} \frac{s_j}{2\lambda_j^2} + b. \qquad (2\text{-}20)$$

8. Simulate $\lambda_j^2 | . \sim$ Inverse Gamma distribution $(1 + \delta, \sigma s_j / 2 + \tau).$

9. Simulate $\tau | . \sim$ Gamma distribution $(k\delta, \sum_{i=1}^{k} \lambda_j^{-2}).$

10. Simulate $\delta|.\sim$ The conditional posterior distribution of δ is

$$P(\delta\,|\,.)=\Gamma(\delta)^{-k}\;\tau^{k\delta}\;\prod_{j=0}^{k}\lambda_j^{-2\delta} \tag{2.21}$$

The Equation (2-21) can be rewritten as

$$P(\delta\,|\,.)=\exp(-k(\ln(\Gamma(\delta))-\delta\ln(\tau))\exp(-2\delta\sum_{j=0}^{k}\ln(\lambda_j)). \tag{2.22}$$

The conditional posterior distribution of δ can be illustrated by the two methods as follows:

3. NEWSUGGESTED OF APPROXIMATION METHODS

In this study, two approximation methods are used to solve the conditional posterior distribution as follows.

1. The first method based on Gamma function. Using the following form for the Gamma function

$$\Gamma(\delta)=\frac{\exp-(\gamma\,\delta)}{\delta}\prod_{L=1}^{\infty}(1+\frac{\delta}{L})^{-1}\exp(\frac{\delta}{L}) \tag{3.1}$$

Where L is integer number, $\gamma=0.577216$ is the Euler-Mascheroni constant see [23].

By substituting Equation (3-1) in to Equation (2-22) the posterior function becomes to

$$P(\delta\,|\,.)=\delta^{k}\,\exp\{k\gamma\delta+k\delta\ln(\tau)+k\sum_{L=1}^{\infty}\left(\ln(1+\frac{\delta}{L})-\frac{\delta}{L}\right)-2\delta\sum_{j=1}^{k}\ln(\lambda_j)\}, \tag{3.2}$$

Under the condition $\delta<1,$ the terms of order three (i.e., $O(\delta^3)$) can be ignored. So the $P(\delta\,|\,.)$ take the following form:

$$P(\delta\,|\,.)\propto\delta^{k}\,\exp\{-\delta\left[2\sum_{j=1}^{k}\ln(\lambda_j)-k\gamma-k\ln(\tau)\right]\}$$
$$\times\exp-\delta^2 k\sum_{L=1}^{\infty}\frac{1}{2L^2}. \tag{3.3}$$

it is known that $\sum_{L=1}^{\infty}\frac{1}{L^2}=\frac{\pi^2}{6}$, see [22]. After some calculation, the Equation (3.3) becomes:

$$P(\delta\,|\,.)\propto\frac{\left[2\sum_{j=1}^{k}\ln(\lambda_j)-k\gamma-k\ln(\tau)\right]^{k+1}}{\Gamma(k+1)}\sqrt{\frac{12}{k\pi^2}}\times$$
$$\delta^{k}\,\exp\{-\delta\left[2\sum_{j=1}^{k}\ln(\lambda_j)-k\gamma-k\ln(\tau)\right]\}\times\exp(-\delta^2\frac{k\pi^2}{12}). \tag{3.4}$$

Then, the posterior density function for δ can be written as:

$$P(\delta\,|\,.)\propto Gamma(k+1,2\sum_{j=1}^{k}\ln(\lambda_j)-k\gamma-k\ln(\tau))\times Normal(0,\frac{12}{k\pi^2}). \tag{3.5}$$

2. The second approximation method is based importance sampling.

Importance sampling technique is a popular sampling tool used for Monte Carlo computing. It is used for evaluating numerical approximation of integrals and it is viewed as a variance reduction technique. Recently, importance sampling is used in variety of applications. In Bayesian analysis, the importance sampling algorithm is used as an approximation to the posterior density for generating random draws. Bayesian computations require us to evaluate

$$E(\phi(\delta)\,|\,\mathrm{x})=\frac{\int_{\delta}\phi(\delta)P(\delta\,|\,\mathrm{x})\,d\delta}{\int_{\delta}P(\delta\,|\,\mathrm{x})\,d\delta} \tag{3.6}$$

where, $P(\delta\,|\,\mathrm{x})$ is the posterior density. Then, evaluating expectations over the posterior distribution requires computing a posterior distribution and often a multidimensional integration. The expectation $E(\phi(\delta)\,|\,\mathrm{x})$ can be approximated using an importance sampling. In its general form, importance sampling approximates the expectation by using a set of samples from some substitute distribution $q(\delta)$ and assigning those samples weights proportional to the ratio $\frac{P(\delta|\mathrm{x})}{q(\delta)}$ (which is related to the normalization factor). Thus, importance sampling provides a simple and efficient way to conduct Bayesian inference via approximating the posterior distribution with samples from the prior distribution weighted by the likelihood. For more details in importance sampling technique see: [24,25].

The importance sampling is proposed to compute the Bayes estimates. The posterior density function for δ, given the data in Equation (3-2), can be reformulated as:

$$P(\delta\,|\,.) = \delta^k \exp\{k\gamma\delta + k\delta\ln(\tau) + k\sum_{L=1}^{\infty}\sum_{m=2}^{\infty}\left(\frac{(-1)^{m+1}(\frac{\delta}{L})^m}{m}\right) - 2\delta\sum_{j=1}^{k}\ln(\lambda_j)\} \quad (3.7)$$

The above equation can be rewritten as

$$P(\delta\,|\,.) \propto \frac{\left[2\sum_{j=1}^{k}\ln(\lambda_j) - k\gamma - k\ln(\tau)\right]^{k+1}}{\Gamma(k+1)}\delta^k \times \exp\{-\delta\left[2\sum_{j=1}^{k}\ln(\lambda_j) - k\gamma - k\ln(\tau)\right]\}\times \exp\{k\sum_{L=1}^{\infty}\sum_{m=2}^{\infty}\left(\frac{(-1)^{m+1}(\frac{\delta}{L})^m}{m}\right)\}. \quad (3.8)$$

Where k is the number of parameters. Then, the posterior density function for δ can be considered as

$$P(\delta\,|\,.) \propto f_{Ga}(\delta, k+1, 2\sum_{j=1}^{k}\ln(\lambda_j) - k\gamma - k\ln(\tau)) \times g(\delta,k), \quad (3.9)$$

where

$$g(\delta,k) \propto \exp\{k\sum_{L=1}^{\infty}\sum_{m=2}^{\infty}\left(\frac{(-1)^{m+1}(\frac{\delta}{L})^m}{m}\right)\}. \quad (3.10)$$

Consider that right-hand side of Equation (3-9) is denoted as $P_N(\delta\,|\,.)$, then, $P_N(\delta\,|\,.)$ and $P(\delta\,|\,.)$ differ only by the proportionality constant. The importance sampling procedure is used to compute the Bayes estimates of any $\phi(\delta)$ using Equation (3-9). The, Bayes estimate of $\phi(\delta)$ under squared error is

$$\hat{\phi}(\delta) = \frac{\sum_{j=1}^{n}\phi(\delta_j)g(\delta_j,k)}{\sum_{j=1}^{n}g(\delta_j,k)}. \quad (3.11)$$

4. SIMULATION STUDY AND CONCLUSION

A simulation study has been made to evaluate the performance of the proposed estimators based on the mean squared error (MSE) criterion and the relative mean squared (RMSE). The evaluation has been done by comparing the MSE and RMSE of the proposed estimators with other well-known estimators. It is noted that the posterior distribution doesn't exist in closed form so the Gibbs sampler method is used to solve our problem. Two approximated methods to solve the full conditional posterior distribution of parameters are suggested. The first method is the approximation method based on the Gamma function. The second method is based on the importance of sample method. Compare and check the accuracy by finding the MSE and RMSE, applying each of than considering different sample sizes and specified distribution problems.

4.1 Simulation Study

The simulation setup is similar to the simulation studies in [17,18] with different parameter values for the error distributions. The quantile regression model used as: $y = x'_i\beta + \varepsilon_i$, where the true value for the $\beta's$ are set as $(3,1.5,0,0,2,0,0,0)'$, where $x_i's$is generated from normal distribution (0,1) during the simulation study. Where ε_i were generated from different distributions with different parameters, so as to explain the influence of the change in the error distributions on the quantile regression equation, which is the basis for choosing through the Bayesian approaches.

For each error distribution with shape parameters and for each sample size, the estimates of β_j where j = 1,…,8 where measured by two the Bayesian approaches.

This paper introduced a program by using Mathcad 15 statistical package to calculate the Bayesian approaches depending on the calculation of the MSE's and RMSE's for the quantile regression parameters, for the two Bayesian methods under consideration.

Random samples (small sample) of size n=15, n=20 and 30 are used. In Bayesian approach, the random samples are generated under the assumption that they are independent and dependent variables.

The distributions have been generated using the following parameters and their parameters respectively

Lognormal ~ log (0,0.6);
Cauchy ~C (0,0.5);
Chi-Square ~ (3).

for each simulation study and for each p ∈ (0.1; 0.25 ; 0.95), where p is represented quantile values which are arbitrary chosen in this study.

These distributions have been generated using the above parameters that were chosen arbitrarily to compare between the methods under considerations with different small sample sizes.

In this paper the mean square errors and relative mean square errors are used as a criteria to compare between the two Bayesian approaches.

The MSE of the estimators $\widehat{\beta}_j$ where j = 1,…, 8 are used to measure its performance, where the MSE's of the estimators $\widehat{\beta}_j$ for the parameters β_j calculated from the sample for β for each parameter. The study repeated samples R = 1000 runs, where R is the number of repeated samples and ($\widehat{\beta}$) are the estimates. The criteria to compare between methods under considerations depend on the approach, which produce a small MSE and small RMSE, for all parameters then it, would be considered more suitable when the objective is to select the variables and estimate the parameters.

The sampling runsR= 1000 replications for each distribution with parameters and three different small sample sizes to be sure of consistency of the results.

For all small sample sizes, for two approaches, and for three distributions parameters, the MSE's for each parameter β_j were calculated using each approach separately.

-All MSE's of Bayesian approximation estimators and Bayesian importance sample estimators presented in the Tables (1) - (9) with small sample sizes (15, 20, 30). We applied these approach in two cases . the first case of Bayesian approach when the random errors are independent importance distribution (i.i.d). to demonstrate the performance of the methods under consideration. .

-In second case of Bayesian approach when the random errors are non- i.i.d. to demonstrate the performance of the methods under consideration.

The data was generated from model of [18],

$$y = 1 + X_1 + X_2 + X_3 + (1 + X_3)\varepsilon$$

where:

$X_1 \sim$ N(0; 1); $X_3 \sim$ uniform(0; 1); $X_2 = X_1 + X_3 + z$ where z ~N(1; 0) and $\varepsilon \sim$ N(0; 1):

Under consideration of two methods, namely approximation methods (based on Gamma distribution and importance method).

The results explored in Tables (1)-(9).

5. DISCUSSION

The aim of this section is to discuss the results two methods with different distributions, different small sample size and different values of quantile. These results presented in Tables (1) - (9). The distributions are log normal, Cauchy and Chi-Square.

In all cases, MSE and RMSE are used to compare among of these methods under consideration. The Gibbs sampler is used to estimate the parameters of the three distribution in Bayesian method. The Bayesian methods, there exist two cases dependent and independent variables and two different methods, approximation and importance sample. By comparison between the results for the two independent methods (IIS and IA) for the small samplesizes (n=15, n=20, n=30) are almost the same. In addition the results for the two dependent methods (DIS and DA) for the sample sizes (n=15, n=20, n=30) again are almost the same. Also MSE and RMSE for the DIS and DA is less than MSE and RMSE of IIS and IA as observed in Tables (1) and (9). The following Tables (1) - (9) presents and summarizes the simulation results based on 1000 repetitions with n=15, n=20 and n=30 for different values of quantile parameter = (0.1. 0.25, 0.95). Also all tables represents the considered four distributions (Cauchy, log normal and Chi-square distributions). The notations in columns IIS, IA, DIS, DA, represents the independent importance sample, independent approximation, dependent importance sample and dependent approximation, respectively. The notations in Rows ER, RER represents estimated risk and relative estimated risk respectively (noted that ER, RER denoted MSE andRMSE).

The results of each distribution for all methods is as follows.

Cauchy distribution: The study determined that when samples (n=15, n=20, n=30) with parameters (0,0.5) for the Cauchy distribution, from the results in Tables (1-3) we see that the results approximately the same for all three sample sizes. The results of MSE and RMSE are almost the same that means the suggested approach is good when the sample size is small.

Chi-Square: From Tables (4) to (6) represented the Chi-Square distribution with degree of freedom equal 3. It is noted that the results for the two dependent methods (DIS and DA) for the sample sizes (n=15, n=20, n=30) are almost the same. Also MSE and RMSE for the DIS and DA is less than MSE and RMSE of IIS and IA.

-Log normal distribution: A comparison between the results for the two independent methods (IIS and IA) for the sample sizes (n=15, n=20, n=30) are almost the same. In addition the results for the two dependent methods (DIS and DA) for the sample sizes (n=15, n=20, n=30) again are almost the same. Also MSE and RMSE for the DIS and DA is less than MSE and RMSE of IIS and IA as observed in Tables (7) and (9).

Table 1. Estimated risks of the Bayesian method for sample size n=15, Cauchy (0, 0.5)

			IIS	**IA**	**DIS**	**DA**
$\rho = 0.1$	ER	β_1	0.011	0.012	0.018	0.017
		β_2	0.034	0.035	0.018	0.017
		β_5	0.027	0.026	0.017	0.018
	RER	β_1	2.694×10^{-4}	6.398×10^{-4}	3.882×10^{-4}	1.064×10^{-5}
		β_2	4.181×10^{-4}	$3.72^{*}\times10^{-4}$	8.345×10^{-4}	1.476×10^{-4}
		β_5	7.917×10^{-4}	6.864×10^{-5}	1.262×10^{-4}	3.447×10^{-5}
$\rho = 0.25$	ER	β_1	0.023	0.025	0.037	0.035
		β_2	0.07	0.073	0.037	0.036
		β_5	0.056	0.053	0.036	0.037
	RER	β_1	5.614×10^{-4}	1.333×10^{-3}	8.087×10^{-4}	2.229×10^{-5}
		β_2	8.714×10^{-4}	7.754×10^{-4}	1.739×10^{-3}	3.074×10^{-4}
		β_5	1.649×10^{-3}	1.429×10^{-4}	2.629×10^{-4}	7.193×10^{-5}
$\rho = 0.9\ 5$	ER	β_1	5.908×10^{-3}	6.394×10^{-3}	9.312×10^{-3}	8.937×10^{-3}
		β_2	0.018	0.019	9.267×10^{-3}	9.19×10^{-3}
		β_5	0.014	0.014	9.124×10^{-3}	9.267×10^{-3}
	RER	β_1	1.422×10^{-4}	3.378×10^{-4}	2.049×10^{-4}	5.588×10^{-6}
		β_2	2.207×10^{-3}	1.966×10^{-4}	4.404×10^{-4}	7.815×10^{-5}
		β_5	4.178×10^{-4}	3.612×10^{-5}	6.658×10^{-5}	1.817×10^{-5}

Where ρ is the quantile chosen

Table 2. Estimated risks of the Bayesian method for sample size n=20, Cauchy (0, 0.5)

			IIS	**IA**	**DIS**	**DA**
$\rho = 0.1$	ER	β_1	0.014	7.126×10^{-3}	0.02	0.02
		β_2	0.041	0.022	0.02	0.02
		β_5	0.031	0.016	0.02	0.02
	RER	β_1	2.854×10^{-4}	1.314×10^{-4}	7.174×10^{-5}	7.144×10^{-5}
		β_2	5.087×10^{-4}	5.456×10^{-4}	2.634×10^{-5}	1.813×10^{-4}
		β_5	1.322×10^{-4}	4.166×10^{-5}	6.654×10^{-4}	2.665×10^{-4}
$\rho = 0.25$	ER	β_1	0.028	0.018	0.042	0.042
		β_2	0.085	0.055	0.042	0.041
		β_5	0.064	0.04	0.042	0.041
	RER	β_1	5.944×10^{-4}	3.285×10^{-4}	1.494×10^{-4}	1.486×10^{-4}
		β_2	1.06×10^{-3}	1.364×10^{-3}	5.491×10^{-5}	3.774×10^{-4}
		β_5	2.755×10^{-3}	1.041×10^{-4}	1.386×10^{-3}	5.555×10^{-4}
$\rho = 0.9\ 5$	ER	β_1	7.131×10^{-3}	0.068	0.011	0.011
		β_2	0.022	0.208	0.011	0.01
		β_5	0.016	0.15	0.011	0.01
	RER	β_1	1.506×10^{-4}	1.249×10^{-3}	3.784×10^{-5}	3.771×10^{-5}
		β_2	2.686×10^{-4}	5.181×10^{-3}	1.392×10^{-5}	9.569×10^{-5}
		β_5	6.979×10^{-4}	3.97×10^{-4}	3.512×10^{-4}	1.406×10^{-4}

Table3. Estimated risks of the Bayesian method for sample size n=30, Cauchy (0, 0.5)

			IIS	IA	DIS	DA
$\rho = 0.1$	ER	β_1	0.017	0.016	0.025	0.025
		β_2	0.048	0.05	0.025	0.023
		β_5	0.038	0.038	0.025	0.025
	RER	β_1	6.612×10^{-5}	6.885×10^{-5}	3.583×10^{-4}	9.71×10^{-4}
		β_2	8.191×10^{-4}	6.113×10^{-4}	9.676×10^{-4}	9.217×10^{-4}
		β_5	1.492×10^{-3}	6.801×10^{-4}	2.124×10^{-3}	4.666×10^{-5}
$\rho = 0.25$	ER	β_1	0.036	0.034	0.052	0.052
		β_2	0.1	0.104	0.052	0.049
		β_5	0.079	0.08	0.053	0.052
	RER	β_1	1.418×10^{-4}	1.434×10^{-4}	7.465×10^{-4}	2.023×10^{-4}
		β_2	1.718×10^{-3}	1.274×10^{-3}	2.016×10^{-3}	1.92×10^{-4}
		β_5	3.099×10^{-3}	1.417×10^{-3}	4.424×10^{-3}	9.715×10^{-5}
$\rho = 0.9\ 5$	ER	β_1	9.214×10^{-3}	8.583×10^{-3}	0.013	0.013
		β_2	0.025	0.026	0.013	0.012
		β_5	0.02	0.02	0.013	0.013
	RER	β_1	3.491×10^{-5}	3.634×10^{-5}	1.891×10^{-4}	5.125×10^{-4}
		β_2	4.324×10^{-4}	3.226×10^{-4}	5.106×10^{-4}	4.865×10^{-4}
		β_5	7.875×10^{-4}	3.589×10^{-4}	1.121×10^{-3}	2.46×10^{-5}

Table 4. Estimated risks of the Bayesian method for sample size n=15, chi square (0, 3)

			IIS	IA	DIS	DA
$\rho = 0.1$	ER	β_1	0.012	0.012	0.018	0.017
		β_2	0.034	0.034	0.018	0.018
		β_5	0.027	0.027	0.018	0.017
	RER	β_1	9.352×10^{-5}	8.945×10^{-5}	9.252×10^{-4}	1.053×10^{-4}
		β_2	8.808×10^{-4}	4.468×10^{-4}	1.861×10^{-4}	2.211×10^{-4}
		β_5	1.156×10^{-3}	2.185×10^{-4}	7.94×10^{-4}	6.091×10^{-4}
$\rho = 0.25$	ER	β_1	0.024	0.025	0.037	0.036
		β_2	0.071	0.072	0.036	0.037
		β_5	0.055	0.056	0.037	0.036
	RER	β_1	1.948×10^{-4}	1.864×10^{-4}	1.927×10^{-3}	2.195×10^{-4}
		β_2	1.835×10^{-3}	9.309×10^{-4}	3.877×10^{-4}	4.607×10^{-4}
		β_5	2.407×10^{-3}	4.553×10^{-4}	1.654×10^{-4}	1.269×10^{-3}
$\rho = 0.9\ 5$	ER	β_1	6.071×10^{-3}	6.254×10^{-3}	9.306×10^{-3}	9.189×10^{-3}
		β_2	0.018	0.018	9.247×10^{-3}	9.302×10^{-3}
		β_5	0.014	0.014	9.351×10^{-3}	9.059×10^{-3}
	RER	β_1	4.936×10^{-5}	4.722×10^{-5}	4.883×10^{-4}	5.561×10^{-5}
		β_2	4.649×10^{-4}	2.359×10^{-4}	9.823×10^{-5}	1.167×10^{-4}
		β_5	6.099×10^{-4}	1.154×10^{-4}	4.19×10^{-5}	3.215×10^{-4}

Table 5. Estimated risks of the Bayesian method for sample size n=20, chi square (0, 3)

			IIS	IA	DIS	DA
$\rho = 0.1$	ER	β_1	0.013	0.013	0.02	0.019
		β_2	0.042	0.039	0.019	0.02
		β_5	0.03	0.031	0.021	0.021
	RER	β_1	2.813×10^{-4}	1.299×10^{-4}	8.248×10^{-4}	3.938×10^{-4}
		β_2	3.56×10^{-3}	8.741×10^{-4}	3.525×10^{-4}	5.886×10^{-4}
		β_5	2.522×10^{-3}	1.642×10^{-3}	1.271×10^{-4}	4.661×10^{-4}

			IIS	IA	DIS	DA
$\rho = 0.25$	ER	β_1	0.028	0.028	0.042	0.04
		β_2	0.087	0.081	0.04	0.042
		β_5	0.062	0.064	0.043	0.043
	RER	β_1	5.86×10^{-4}	2.707×10^{-4}	1.718×10^{-3}	8.205×10^{-4}
		β_2	7.416×10^{-4}	1.821×10^{-3}	7.344×10^{-4}	1.226×10^{-3}
		β_5	5.255×10^{-4}	3.421×10^{-3}	2.648×10^{-4}	9.712×10^{-4}
$\rho = 0.95$	ER	β_1	7.112×10^{-3}	6.989×10^{-3}	0.011	0.01
		β_2	0.022	0.021	0.01	0.011
		β_5	0.016	0.016	0.011	0.011
	RER	β_1	1.484×10^{-4}	6.854×10^{-4}	4.353×10^{-4}	2.078×10^{-4}
		β_2	1.879×10^{-3}	4.614×10^{-4}	1.86×10^{-4}	3.107×10^{-4}
		β_5	1.331×10^{-3}	8.668×10^{-4}	6.71×10^{-5}	2.46×10^{-4}

Table 6. Estimated risks of the Bayesian method for sample size n=30, chi square (0, 3)

			IIS	IA	DIS	DA
$\rho = 0.1$	ER	β_1	0.017	0.016	0.025	0.024
		β_2	0.048	0.048	0.025	0.024
		β_5	0.038	0.038	0.026	0.024
	RER	β_1	1.618×10^{-4}	7.81×10^{-4}	6.227×10^{-5}	3.557×10^{-4}
		β_2	8.184×10^{-4}	2.177×10^{-3}	5.828×10^{-4}	1.901×10^{-4}
		β_5	9.23×10^{-4}	1.688×10^{-4}	1.848×10^{-5}	1.59×10^{-3}
$\rho = 0.25$	ER	β_1	0.035	0.034	0.052	0.05
		β_2	0.101	0.099	0.051	0.05
		β_5	0.079	0.078	0.053	0.051
	RER	β_1	3.37×10^{-4}	1.627×10^{-3}	1.298×10^{-4}	7.411×10^{-4}
		β_2	1.705×10^{-3}	4.535×10^{-3}	1.214×10^{-3}	3.96×10^{-4}
		β_5	1.923×10^{-3}	3.513×10^{-4}	3.852×10^{-5}	3.314×10^{-3}
$\rho = 0.95$	ER	β_1	8.986×10^{-3}	8.677×10^{-3}	0.013	0.013
		β_2	0.026	0.025	0.013	0.013
		β_5	0.02	0.02	0.014	0.013
	RER	β_1	8.539×10^{-5}	4.122×10^{-4}	3.286×10^{-5}	1.877×10^{-4}
		β_2	4.32×10^{-4}	1.149×10^{-3}	3.076×10^{-4}	1.003×10^{-4}
		β_5	4.871×10^{-3}	8.906×10^{-5}	9.748×10^{-6}	8.394×10^{-4}

Table 7. Estimated risks of the Bayesian method for sample size n=15, log normal (0,0.6)

			IIS	IA	DIS	DA
$\rho = 0.1$	ER	β_1	0.011	0.012	0.018	0.017
		β_2	0.034	0.035	0.018	0.018
		β_5	0.026	0.026	0.017	0.018
	RER	β_1	1.838×10^{-4}	4.345×10^{-4}	3.882×10^{-4}	3.324×10^{-4}
		β_2	7.08×10^{-4}	1.404×10^{-3}	8.345×10^{-4}	6.37×10^{-4}
		β_5	1.556×10^{-4}	5.88×10^{-4}	1.262×10^{-4}	5.976×10^{-4}
$\rho = 0.25$	ER	β_1	0.024	0.024	0.037	0.036
		β_2	0.071	0.072	0.037	0.037
		β_5	0.054	0.054	0.036	0.037
	RER	β_1	3.829×10^{-4}	9.052×10^{-4}	8.087×10^{-4}	6.923×10^{-4}
		β_2	1.475×10^{-3}	2.928×10^{-3}	1.739×10^{-3}	1.327×10^{-3}
		β_5	3.243×10^{-4}	1.226×10^{-3}	2.628×10^{-4}	1.245×10^{-3}
$\rho = 0.95$	ER	β_1	6.05×10^{-3}	6.147×10^{-3}	9.312×10^{-3}	9.145×10^{-3}
		β_2	0.018	0.018	9.267×10^{-3}	9.408×10^{-3}
		β_5	0.014	0.014	9.124×10^{-3}	9.447×10^{-3}
	RER	β_1	9.702×10^{-5}	2.293×10^{-4}	2.049×10^{-4}	1.754×10^{-4}
		β_2	3.737×10^{-4}	7.411×10^{-4}	4.404×10^{-4}	3.362×10^{-4}
		β_5	8.212×10^{-5}	3.103×10^{-4}	6.659×10^{-5}	3.154×10^{-4}

Table 8. Estimated risks of the Bayesian method for sample size n=20, log normal (0,0.6)

			IIS	IA	DIS	DA
$\rho = 0.1$	ER	β_1	0.013	0.013	0.02	0.02
		β_2	0.04	0.04	0.02	0.02
		β_5	0.03	0.031	0.02	0.02
	RER	β_1	7.362×10^{-6}	5.36×10^{-4}	6.02×10^{-4}	2.307×10^{-6}
		β_2	1.855×10^{-5}	8.505×10^{-4}	7.745×10^{-4}	7.149×10^{-5}
		β_5	9.787×10^{-5}	3.896×10^{-4}	4.048×10^{-4}	1.002×10^{-3}
$\rho = 0.25$	ER	β_1	0.028	0.028	0.041	0.042
		β_2	0.083	0.083	0.041	0.041
		β_5	0.062	0.065	0.041	0.042
	RER	β_1	1.543×10^{-5}	1.116×10^{-3}	1.254×10^{-3}	4.853×10^{-6}
		β_2	3.861×10^{-4}	1.771×10^{-3}	1.613×10^{-3}	1.489×10^{-4}
		β_5	2.037×10^{-4}	8.113×10^{-4}	8.433×10^{-4}	2.089×10^{-3}
	ER	β_1	7.086×10^{-3}	7.106×10^{-3}	0.01	0.011
		β_2	0.021	0.021	0.01	0.01
		β_5	0.016	0.017	0.01	0.011
	RER	β_1	3.887×10^{-6}	2.829×10^{-4}	3.177×10^{-4}	1.218×10^{-6}
		β_2	9.79×10^{-5}	4.491×10^{-4}	4.087×10^{-4}	3.773×10^{-5}
		β_5	5.165×10^{-5}	2.057×10^{-4}	2.136×10^{-4}	5.291×10^{-4}

Table 9. Estimated risks of the Bayesian method for sample size n=30, log normal (0,0.6)

			IIS	IA	DIS	DA
$\rho = 0.1$	ER	β_1	0.016	0.017	0.024	0.025
		β_2	0.047	0.048	0.025	0.025
		β_5	0.036	0.037	0.025	0.025
	RER	β_1	4.231×10^{-4}	1.135×10^{-5}	9.973×10^{-4}	2.356×10^{-4}
		β_2	4.735×10^{-3}	7.515×10^{-4}	7.164×10^{-4}	5.41×10^{-4}
		β_5	3.067×10^{-4}	2.327×10^{-3}	4.632×10^{-4}	1.671×10^{-4}
$\rho = 0.25$	ER	β_1	0.034	0.036	0.05	0.053
		β_2	0.099	0.101	0.052	0.051
		β_5	0.076	0.076	0.051	0.052
	RER	β_1	8.831×10^{-4}	2.367×10^{-5}	2.078×10^{-3}	4.908×10^{-4}
		β_2	9.858×10^{-3}	1.565×10^{-3}	1.493×10^{-3}	1.127×10^{-3}
		β_5	6.351×10^{-4}	4.848×10^{-3}	9.652×10^{-4}	3.481×10^{-4}
$\rho = 0.95$	ER	β_1	8.572*10^-3	8.996×10^{-3}	0.013	0.013
		β_2	0.025	0.026	0.013	0.013
		β_5	0.019	0.019	0.013	0.013
	RER	β_1	2.233×10^{-4}	5.991×10^{-6}	5.264×10^{-4}	1.243×10^{-4}
		β_2	2.499×10^{-3}	3.966×10^{-4}	3.781×10^{-4}	2.856×10^{-4}
		β_5	1.618×10^{-4}	1.228×10^{-3}	2.445×10^{-4}	8.818×10^{-5}

In general, in the above simulation study for all values of the parameter □, the results shows that in the independent approximation method and the importance sample same behavior a small sample sizes observed in all tables. The mean square error and relative mean square error decreases as the sample size (n) increases for all different distributions and different values of the quantile parameter (□) in two methods. By comparison between the results for the three distributions, we found that approximate the same behavior for the small sample size.

6. CONCLUSION

In the above sections treatment the quantile regression model, Bayesian adaptive Lasso quantile regression (BALQR) by using a small sample size. Also they put different penalization parameters on the different regression coefficients. Bayesian adaptive Lasso quantile regression approaches used to select the best subset of variables and estimate the parameters of the quantile regression equation when small sample sizes are used. The posterior distribution doesn't exist in the closed form so Gibbs sampler method is used to solve the our problem. The full conditional posterior distribution is solved by using the new approaches approximation method and importance sample method. The numerical results of MSE and RMSE for the three distributions (Cauchy, Chi-Square, Log-Normal distributions) for many values of ρ and small different sample size in two case (approximation method and importance sample). The simulations study shown that the proposed approaches are very competitive in terms of variable selection, estimation accuracy and efficient when small sample sizes are used.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

1. Koenker R, Bassett G. Regression quantiles. Econometrica. 1978;46:33-50.
2. Elham Abdul-Razik Ismail, Neveen Sayed- Ahmed. A comparison between linear regression and lasso quantile regression methods in variables selection. International Journal of current Business and Economics. 2015;4:69-74.
3. Wu Y, Liu Y. Variable selection in quantile regression. Statistica Sinica. 2009;19: 801-817.
4. Belloni A, Chernozhukov V. L1-penalized quantile regression in high-dimensional sparse models. The Annals of Statistics. 2011;39(1):82–130.
5. Zou H, Yuan M. Regularized simultaneous model selection inmultiple quantiles regression. Computational Statistics and Data Analysis. 2008;52: 5296-5304.
6. Yu K, Moyeed RA. Bayesian quantile regression. Statistics and Probability Letters. 2001;54:437-447.
7. Keming Yu, Philippe Van Kerm, Jin Zhang Bayesian quantile regression: An application to the wage distribution in 1990s Britain. Indian Statistical Institute. 2005;67:359-377.
8. Yu K, Stander J. Bayesian analysis of a to bit quantile regression model. Journal of Econometrics. 2007;137:260-276.
9. Geraci M, Bottai M. Quantile regression for longitudinaldata using the asymmetric Laplace distribution. Biostatistics. 2007;8: 140-154.
10. Chen C, Yu K. Automatic Bayesian quantile regression curvefitting. Statistics and Computing. 2009;19(3):271-281.
11. Yang Y, Wang HJ, He X. Posterior inference in Bayesian quantile regression with asymmetric Laplace likelihood. International Statistical Review. 2015; 84(3):327-344.
12. Sriram K. A sandwich likelihood correction for Bayesian quantile regression based on the misspecified asymmetric Laplace density. Statistics & Probability Letters. 2015;107:18-26.
13. Thompson P, Cai Y, Moyeed R, Reeve D, Stander J. Bayesian nonparametric quantile regression using splines. Computational Statistics and Data Analysis. 2010;54(4):1138-1150.
14. Kottas A, Krnjajifc M. Bayesian semiparametric modelingin quantile regression. Scandinavian Journal of Statistics. 2009;36:297-319. Alhamzawi R, Yu K, Benoit T. Bayesian adaptive lassoquantile regression. Statistical Modelling. 2012;12(3):279-297.
15. Benoit DF, Van den Poel D. Bayes QR: A Bayesian approach to quantile regression. Journal of Statistical Software. 2017; 76(7):1-31.
16. Kottas A, Gelfand AE. Bayesian semi-parametric median re-gression model. J. Am. Statist. Ass. 2001;96:1458-1468.
17. Li Q, Xi R, Lin N. Bayesian regularized quantile regression. Bayesian Analysis. 2010;5:1-24.
18. Alhamzawi R, Yu K, Benoit T. Bayesian adaptive lassoquantile regression. Statistical Modelling. 2012;12(3):279-297.
19. Yi N, Xu S. Bayesian lasso for quantitative trait loci mapping. Genetics. 2008;179:1045-1055.
20. Sun W, Ibrahim JG, Zou F. Genomewide multiple-loci mapping in experimental crosses by iterative adaptive penalized regression. Genetics Society of America. 2010;185:349-359.
21. Andrews DF, Mallows CL. Scale mixtures of normal distributions. Journal of the Royal Statistical Society, Ser. B. 1974;36: 99-102.
22. Wilfred Kaplan. Advanced calculus fifth edition. Publishing House of Electronics Industry; 1949.

23. Remmert R, Kay LD. (translator). Classical topics in complex function theory. Springer; 2006. ISBN 0-387-98221-3
24. Kundu D, Howlader H. Bayesian inference and prediction of the inverse We ibull distribution for type-II censored data. Computational Statistics and Data Analysis. 2010;54:1547-1558.
25. Tokdar ST, Kass RE. Importance sampling. A Review, Wiley Inter-disciplinary Reviews: Computational Statistics. 2010;2:54-60.

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