A Note on \(\alpha\)-stable and \(\alpha\)-inverse Gaussian Laws

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Authors’ contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this article we obtain the first passage time distribution of \(\alpha\)-stable Lévy processes. We derive the moment estimators of the parameters of \(\alpha\)-inverse Gaussian laws and also their asymptotic distribution.

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1 Introduction

\(\alpha\)-stable laws are infinitely divisible and hence one can define corresponding \(\alpha\)-stable Lévy processes (\(\alpha\)SLP). The range of \(\alpha\) is \(0 < \alpha \leq 2\) and for \(\alpha = 2\) the \(\alpha\)-stable law is Gaussian/normal law and the corresponding \(\alpha\)SLP is the Brownian motion process (BMP). It is known that \(\frac{1}{2}\)-stable law is the first passage time (FPT) distribution of BMP with zero drift ([1], p.174) and for a BMP with positive drift, the FPT distribution is inverse Gaussian (IG) ([2], p.137). IG laws were generalized to \(\alpha\)-IG (\(\alpha\)IG) laws in [3].

Here, in section 2, we obtain the FPT distribution of \(\alpha\)SLP. Since the density of \(\alpha\)IG laws is not in closed form we derive the moment estimators of its parameters and obtain their asymptotic distribution in section 3. Another possible approach based on the p.d.f. of Gamma is also sketched.

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2 FPT Distribution of $\alpha$SLP

We need the following result from [4] to define the $\alpha$SLP. They call it extreme stable since the parameter $\beta$ in the stable model is set as $\beta = 1$. They have taken the location parameter also as zero. However, here we refer to them as $\alpha$-stable laws.

**Theorem 2.1.** The function $M(s) = \exp\{-b(1-\alpha)s^\alpha\}; 0 \leq \text{Re}(s) < \infty; 0 < \alpha \leq 2, \alpha \neq 1, b > 0$ are moment generating functions (MGF) of $\alpha$-stable laws.

**Definition 2.1.** Lévy processes $\{X(t); t \geq 0\}$ are $\alpha$SLP, if the distribution of $X(1)$ is $\alpha$-stable with MGF $M(s) = \exp\{b(\alpha - 1)s^\alpha\}$.

FPT distributions of processes are important as they give the distribution of the time taken for the process to reach/ cross a barrier. If $\lambda > 0$ is the barrier, then the random variable (r.v.) $T(\lambda) = T = \inf\{t > 0 : X(t) \geq \lambda\}$ denote the FPT of $X(t)$. Here $t > 0$, since $X(0) = 0$ for a Lévy process. Since the location parameter is zero for the $\alpha$-stable laws considered here, the $\alpha$SLP has zero drift. Further, for $1 < \alpha \leq 2$, it has finite mean and hence martingale based arguments on $X(t)$ are justified. We now derive the FPT distribution of $\alpha$SLP using standard arguments based on optimal sampling theorem (see, [5]) applied to the martingale of $\{X(t)\}$ in proposition 2.1. Other relevant literature are: [6, 7, 8, 9, 10].

**Proposition 2.1.** For the $\alpha$SLP $\{X(v), v \geq 0\}$, $W(v) = \exp\{sX(v) - \theta v\}, s > 0$ a constant, is a martingale, where $\theta = b(\alpha - 1)s^\alpha$.

**Proof.** Since, $E\left(e^{sX(t)}\right) = e^{\theta t}, E(|W(v)|) = E(W(v)) = e^{-\theta v}E\left(e^{sX(v)}\right) = 1 < \infty$. Since Lévy processes have stationary and independent increments, for $u \leq v$, $X(v) - X(u)$ is independent of $\mathcal{F}_u$, the filtration up to time $u$. Now,

$$
E\left(W(v) | \mathcal{F}_u\right) = E\left(\exp\{sX(v) - \theta v| \mathcal{F}_u\}\right)
= e^{-\theta v}E\left(e^{s(X(v) - X(u)) + sX(u)}| \mathcal{F}_u\right)
= e^{-\theta v}E\left(e^{s(X(v) - X(u))}| \mathcal{F}_u\right)E\left(e^{sX(u)}| \mathcal{F}_u\right)
= e^{-\theta v}E\left(e^{sX(v-u)}| \mathcal{F}_u\right)E\left(e^{sX(u)}\right)
= e^{-\theta v}e^{\theta (v-u)}E\left(e^{sX(u)}\right) = e^{sX(u) - \theta u} = W(u),
$$

and that completes the proof. \hfill \Box

**Theorem 2.2.** The FPT distribution of $\alpha$SLP for $1 < \alpha \leq 2$, is $\frac{1}{\lambda}$-stable.

**Proof.** Let the r.v. $T(\lambda) = T$ denote the FPT for the $\alpha$SLP $\{X(t), t \geq 0\}$ to reach or cross $\lambda > 0$. We know that for $\{X(t)\}$, $W(t) = \exp\{sX(t) - \theta t\}$ is a martingale, where $\theta = b(\alpha - 1)s^\alpha$. For a martingale $\{W(t)\}$ and for the FPT $T$ (which is a stopping time), $E\{W(0)\} = E\{W(T \wedge t)\}$. As $X(0) = 0, W(0) = 1$ and hence $E\{W(T \wedge t)\} = 1$. That is,

$$
E\left[\exp\{sX(T \wedge t) - \theta (T \wedge t)\}\right] = 1, \quad (2.1)
$$

Note that $\theta = b(\alpha - 1)s^\alpha > 0$ for $1 < \alpha \leq 2$ and so $0 \leq W(T \wedge t) \leq e^{\lambda t}$. Now assuming $P\{T < \infty\} = 1$ (we will justify this at the end of the proof) we may pass to the limit as $t \to \infty$ under the expectation in (2.1) by the optional sampling theorem, yielding;

$$
1 = \lim_{t \to \infty} E\left[\exp\{sX(T \wedge t) - \theta (T \wedge t)\}\right] = e^{\lambda t}E\left[e^{-\theta T}\right].
$$
Thus, \( E[e^{-\theta T}] = e^{-s\lambda} \).

Since \( \theta = b(\alpha - 1)s^\alpha \implies s = \left\{ \frac{a}{b(\alpha - 1)} \right\}^{1/\alpha} \), we get the LT of the FPT as,

\[
E[e^{-\theta T}] = e^{\left[ -\frac{\lambda}{b(\alpha - 1)^{1/\alpha}} \right]^{1/\alpha}},
\]

which is that of \( \frac{1}{\alpha} \)-stable law ([1], p.448).

Finally, since \( P(T < \infty) = \lim_{t \to 0} E[e^{-\theta T}] = 1 \), we get a proper distribution, justifying our assumption \( P(T < \infty) = 1 \).

Remark 2.1. \( E[e^{-\theta T}] = e^{-s\lambda} \) is the LT of a probability distribution only when \( \theta^{1/\alpha} \) has completely monotone derivative. That is, if \( 0 < \frac{1}{\alpha} < 1 \implies \alpha > 1 \) ([1], p.448). Also, here we need \( \theta > 0 \). These are the reasons for restricting the range of \( \alpha \) to \( 1 < \alpha \leq 2 \) in the above theorem. When \( \alpha = 1 \) the distribution is degenerate.

Remark 2.2. One may note that when \( \alpha = 2 \), the SLP is BMP and the LT of \( T \) is \( E[e^{-\theta T}] = e^{\left[ -\frac{\lambda}{2} \right]} \) which is that of \( \frac{1}{2} \)-stable, as is known.

3 Estimation of Parameters of \( \alpha \text{IG} \)

By [3], a r.v. \( X \sim \alpha \text{IG}(\alpha, \mu, m) \) (\( \alpha \text{IG} \)), if its LT is;

\[
L(s) = \exp\left\{ \frac{m}{\mu} [1 - (1 + \frac{2\mu^2}{m})^\alpha] \right\}; s \geq 0, \ 0 < \alpha < 1, \ \mu, \ m > 0.
\]

The probability density function (p.d.f.) of \( X \sim \alpha \text{IG}(\alpha, \mu, m) \) is;

\[
f(x) = \frac{1}{c} \exp\left( \frac{m}{\mu} \left( 1 - \frac{x}{2\mu} \right) \right) p_\alpha\left( \frac{x}{c} \right), \ x > 0, \ \text{where} \ c = 2m^{\frac{1}{\alpha} - 1} \mu^{2 - \frac{1}{\alpha}},
\]

and \( p_\alpha(x) = \frac{1}{\pi x} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(ak + 1)}{k!} \sin(k\pi\alpha)(x^{-\alpha})^k \), \( x > 0 \),

is the p.d.f. of \( \alpha \)-stable law. Since the p.d.f. of \( \alpha \text{IG} \) law is complex, estimation based on p.d.f.s is not easy and so we adapt that in [11].

From the logarithm of the LT of \( \alpha \text{IG} \) we get its \( r \)th cumulant as:

\[
\kappa_r = \left[ \frac{\partial^r \log L(s)}{\partial s^r} \right]_{s=0} = -\frac{m}{\mu} \alpha(\alpha - 1)...(\alpha - r + 1) \left[ -\frac{2\mu^2}{m} \right]^r = -\frac{m}{\mu} (\alpha, r) \left[ -\frac{2\mu^2}{m} \right]^r.
\]

Using the relations ([12], p.101) connecting \( \kappa_j \) and \( \mu_j \) the central moments, (for direct computation see, [3]) we have,

\[
\begin{align*}
\mu_1 &= \kappa_1 = \frac{m}{\mu} \alpha - \frac{2\mu^2}{m} = 2\alpha \mu, \quad (3.1) \\
\mu_2 &= \kappa_2 = \frac{m}{\mu} \alpha(\alpha - 1) \left( -\frac{2\mu^2}{m} \right)^2 = 4\alpha(1 - \alpha) \mu^3 \quad \text{and} \quad (3.2) \\
\mu_3 &= \kappa_3 = \frac{m}{\mu} \alpha(\alpha - 1)(\alpha - 2) \left( -\frac{2\mu^2}{m} \right)^3 = 8\alpha(1 - \alpha)(\alpha - 2) \frac{\mu^5}{m^2}. \quad (3.3)
\end{align*}
\]
From (3.1) and (3.2) \(2\mu_1(1-\alpha)\frac{\mu^2}{m} = \mu_2\), (3.4)  
from (3.2) and (3.3) \(2\mu_2(2-\alpha)\frac{\mu^2}{m} = \mu_3\), and (3.5)  
from (3.4) and (3.5) \(\frac{\mu_1(1-\alpha)}{\mu_2(2-\alpha)} = \frac{\mu_2}{\mu_3}\), (3.6)  

Solving, from (3.6), \(\alpha = \frac{2\mu_2^2 - \mu_2 \mu_3}{\mu_2^2 - \mu_1 \mu_3} = 1 + \frac{\mu_2^2}{\mu_2^2 - \mu_1 \mu_3}\).  
from (3.1), \(\mu = \frac{\mu_1}{2\alpha} = \frac{\mu_1}{2} \left(\frac{\mu_2^2 - \mu_1 \mu_3}{\mu_2^2 - \mu_1 \mu_3}\right)\) and  
from (3.4), \(m = \frac{2\mu_1}{\mu_2}(1-\alpha)\mu^2 = \frac{\mu_1^2}{2} \left(\frac{\mu_1 \mu_3 - \mu_2^2}{\mu_2^2 - \mu_1 \mu_3}\right)^2\).  

Let \(x_1, \ldots, x_n\) be a simple random sample of size \(n\) from \(X \sim \alpha IG(\alpha, \mu, m)\) and \(a, b, c\) respectively denote the mean, variance and third central moment from the sample. Now the moment estimators of the parameters \(\alpha, \mu\) and \(m\) are:  
\[\hat{\alpha} = \frac{2b^2 - ac}{b^2 - ac} = 1 + \frac{b^2}{b^2 - ac}\]  
\[\hat{\mu} = \frac{a}{2\hat{\alpha}} = \frac{a}{2} \left(\frac{b^2 - ac}{2b^2 - ac}\right)\text{ and}\]  
\[\hat{m} = \frac{2a}{b}(1-\alpha)\mu^2 = \frac{a^2b}{2} \left(\frac{ac - b^2}{2b^2 - ac}\right)^2\].  

Remark 3.1. From remark 5 in [3], Gamma\((\alpha, 2/\theta)\) law is the mixture of \(X \sim \alpha IG(\alpha, \mu, m)\) with \(\mu \sim Exp(\theta)\). The corresponding stochastic representation is \(Y = XE\). This opens up the possibility of transforming the observations on \(\alpha IG\) r.v. \(X\) to the corresponding gamma r.v. \(Y\), estimating the parameters of the gamma and in turn that of the \(\alpha IG\). Comparing these estimates with the moment estimates obtained above for their efficiency and closeness is worth investigating. This demands simulation studies and will be reported elsewhere.

3.1 Asymptotic distribution of the estimators

If \(X \sim \alpha IG(\alpha, \mu, m)\), then the vector \((X, X^2, X^3)'\) has mean \(\Sigma = (\alpha_1, \alpha_2, \alpha_3)'\) and covariance matrix  
\[
\Sigma = \begin{bmatrix}
\alpha_2 - \alpha_1^2 & \alpha_3 - \alpha_1 \alpha_2 & \alpha_4 - \alpha_1 \alpha_3 \\
\alpha_3 - \alpha_1 \alpha_2 & \alpha_4 - \alpha_1 \alpha_3 & \alpha_6 - \alpha_1 \alpha_3 \\
\alpha_4 - \alpha_1 \alpha_3 & \alpha_6 - \alpha_1 \alpha_3 & \alpha_6 - \alpha_1 \alpha_3
\end{bmatrix},
\]
where \(\alpha_k = E(X^k)\), \(k = 1, 2, \ldots, 6\). Define \(X_i = \frac{1}{n} \sum_{i=1}^n x_i^j\), \(j = 1, 2, 3\) and put \(\bar{X} = (\bar{X}^1, \bar{X}^2, \bar{X}^3)'\). Then, by the multivariate CLT ([12], p.128),  
\[
\sqrt{n}(\bar{X} - \alpha) \xrightarrow{d} Z_1 \sim N_3(0, \Sigma).
\]

Using the relations between \(\alpha_i\) and \(\mu_j\) ([12], p.101), define the following functions \(g_i\), so that we get \(g_i(\bar{X})\), \(i = 1, 2, 3\), the corresponding moment estimators.  
\[g_1(\alpha) = g_1(\alpha_1, \alpha_2, \alpha_3) = \alpha = 1 + \frac{(\alpha_2 - \alpha_1^2)^2}{(\alpha_2 - \alpha_1^2)^2 - \alpha_1(3\alpha_1 \alpha_2 + 2\alpha_1^2)},\]  
\[g_2(\alpha) = g_2(\alpha_1, \alpha_2, \alpha_3) = \mu = \frac{\alpha_1}{2\alpha},\text{ and}\]  
\[g_3(\alpha) = g_3(\alpha_1, \alpha_2, \alpha_3) = m = 2 - \frac{\alpha_1}{\alpha_2 - \alpha_1^2}(1 - \alpha)\mu^2\].

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Since $g_i$’s are totally differentiable, we get the asymptotic joint distribution of the moment estimators invoking the result (iii) ([12], p.388) as,

\[ \sqrt{n} \left( g_1(X) - g_1(\bar{X}), g_2(X) - g_2(\bar{X}), g_3(X) - g_3(\bar{X}) \right) \xrightarrow{d} Z_2 \sim N_3(0, G\Sigma G'), \]

where $G = \frac{\partial g_i(\bar{x})}{\partial x_j}$. 

[3] showed that $\alpha$IG laws are self-decomposable. This motivated [13] to develop AR(1) models with IG marginals. [14] discussed Bayesian analysis of IG stochastic conditional duration models. [10] derived the FPT distribution of IG process. These suggest the possibility of discussing $\alpha$IG laws in other modelling contexts.

4 Summary

In this paper the FPT distribution of $\alpha$SLP, for $1 < \alpha \leq 2$, is obtained as $\frac{1}{\alpha}$-stable law. Moment estimators of the parameters of $\alpha$IG law are derived and they are shown to be jointly asymptotically normal.

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Competing Interests

Author has declared that no competing interests exist.

References


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